Catalytic approaches to the Tree Evaluation Problem

James Cook, Ian Mertz

STOC 2020

Outline

The Tree Evaluation Problem

New algorithm

Pebbles and Branching Programs for Tree Evaluation [S. Cook, P. McKenzie, D. Wehr, M. Braverman, R. Santhanam 2010]
New Results for Tree Evaluation [S. Chan, J. Cook, S. Cook, P. Nguyen, D. Wehr 2010]

The Tree Evaluation Problem

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- Branching programs and pebbling games
- Lower bounds

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The Tree Evaluation Problem (TEP)

Motivation

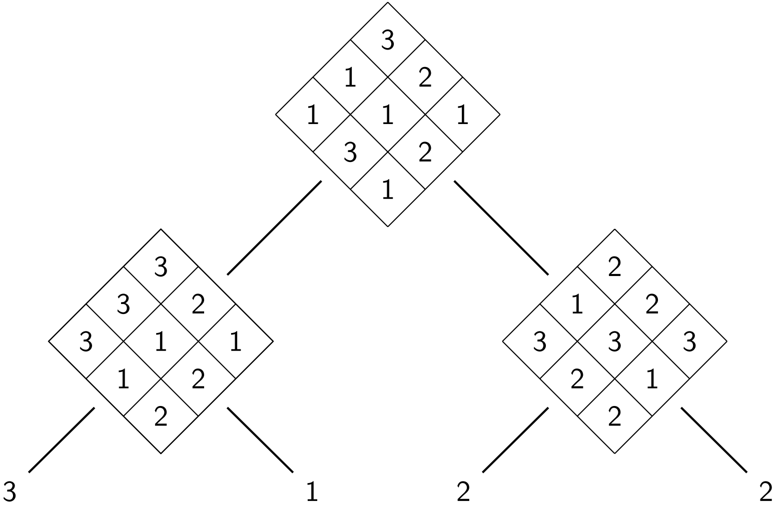
Fact

TEP $\in$ P

Conjecture

TEP $\notin$ L

The Tree Evaluation Problem (TEP)



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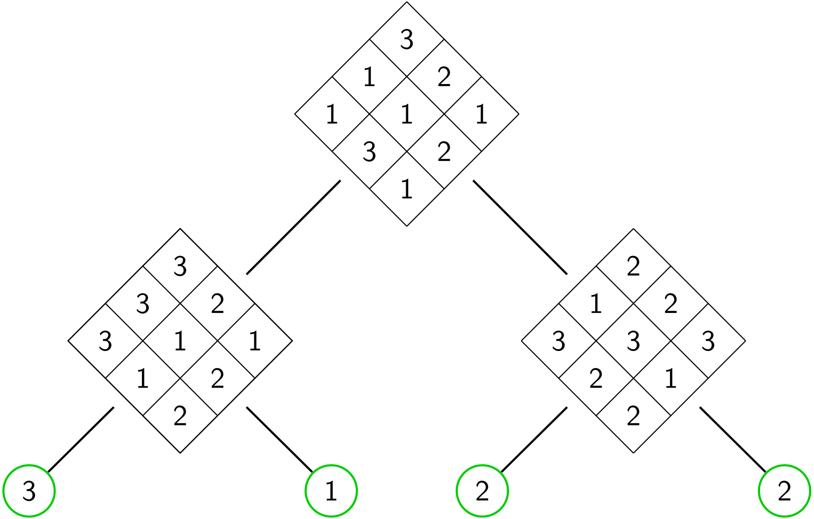

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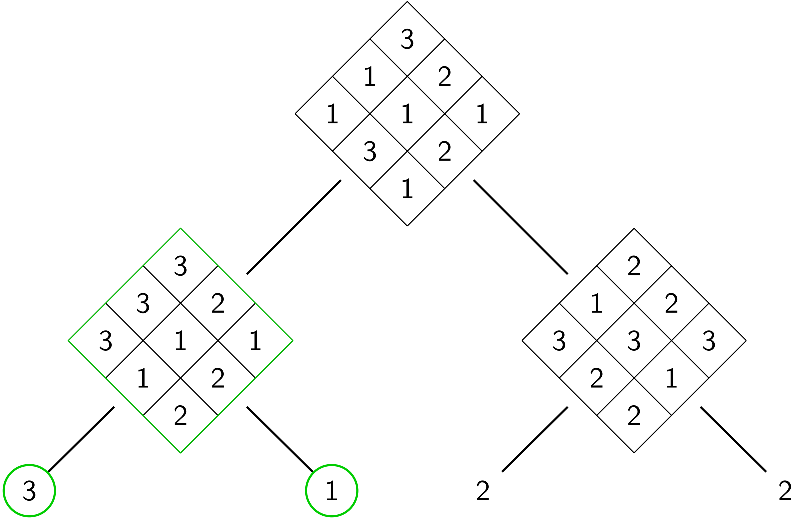

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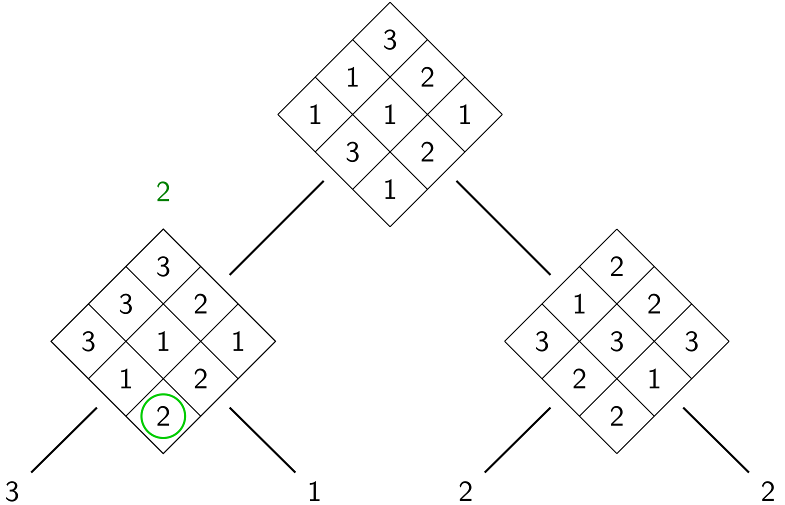

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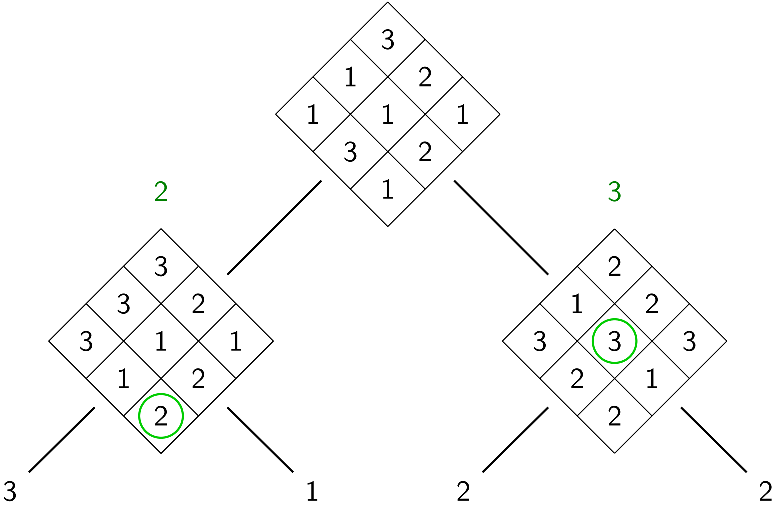

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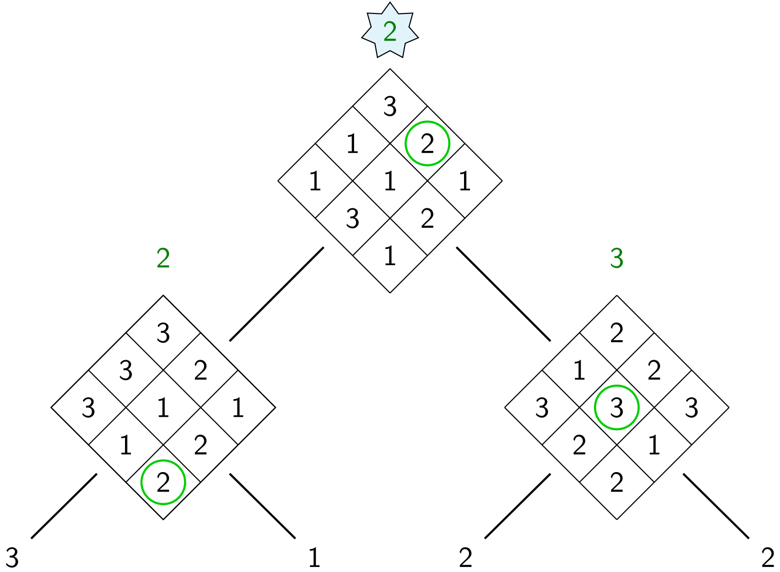

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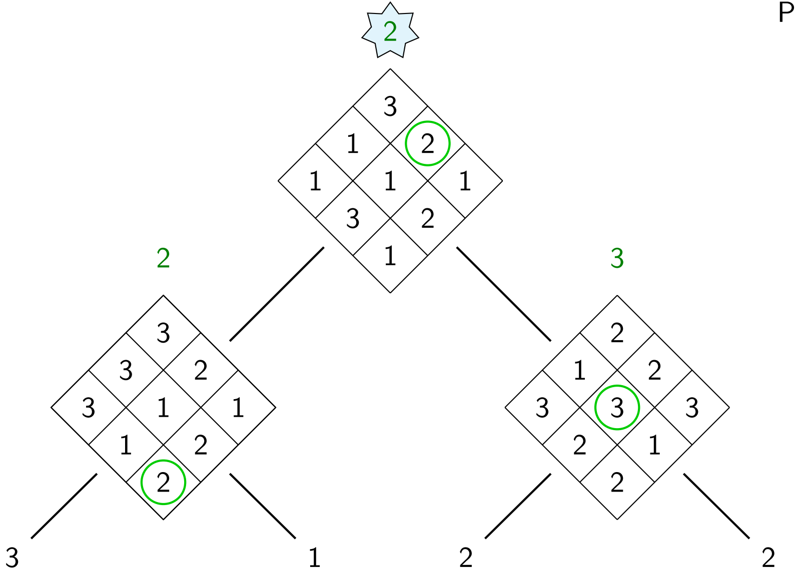

Parameters:

- height = 3
- k = 3

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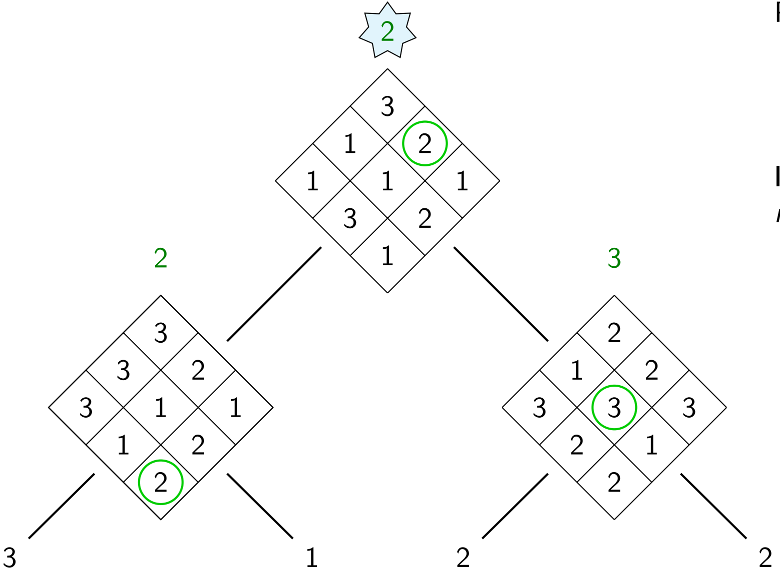

Parameters:

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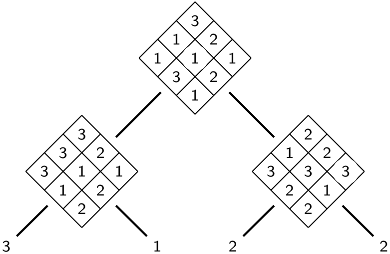
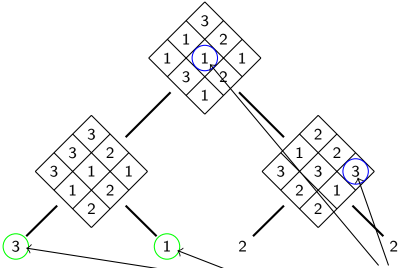
Input size:
$n = \Theta(2^h k^2 \log k)$ bits.

TEP Input size: $\Theta(2^h k^2 \log k)$.

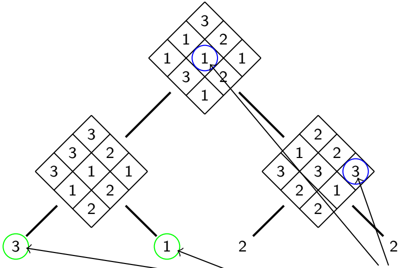
Conjecture

TEP $\notin$ L
In other words, it can't be solved in $O(h + \log k)$ space.


3
1 2
1 1 1
3 2
1
3
3 2
3 1 1
1 2
2
2
1 2
3 3 3
2 1
2
3
1
2
2



A *query* is either a leaf or a cell in a table of an internal node.



A *query* is either a leaf or a cell in a table of an internal node.

A *branching program* is a directed graph of *states*. There are two kinds of state:

- *query state*: labelled with a query and has k outgoing edges labelled with the possible answers.
- *final state*: labelled with a number $1..k$.

One state is the starting state.

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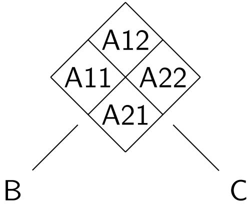
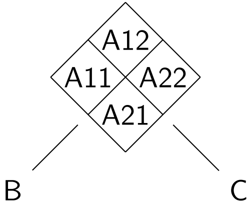

A12

A11 A22

A21

B C


A12
A11
A22
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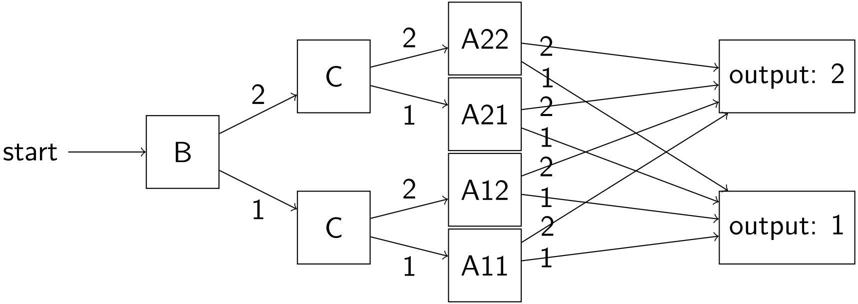
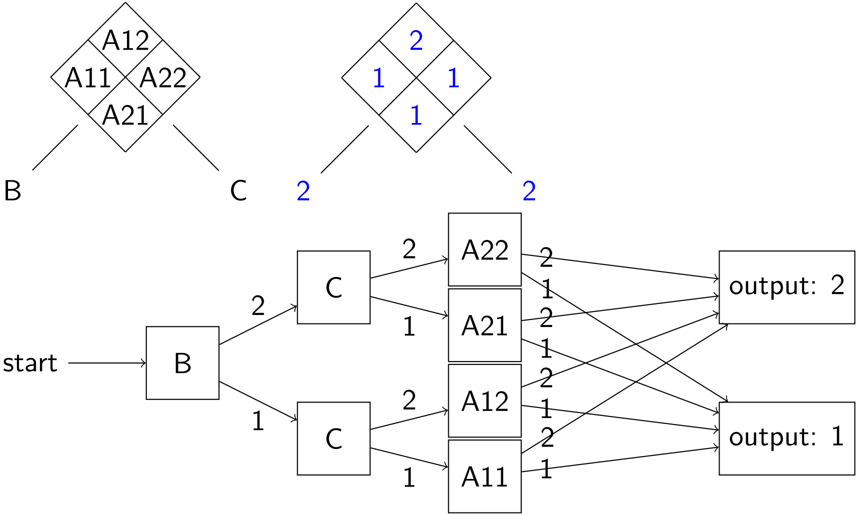
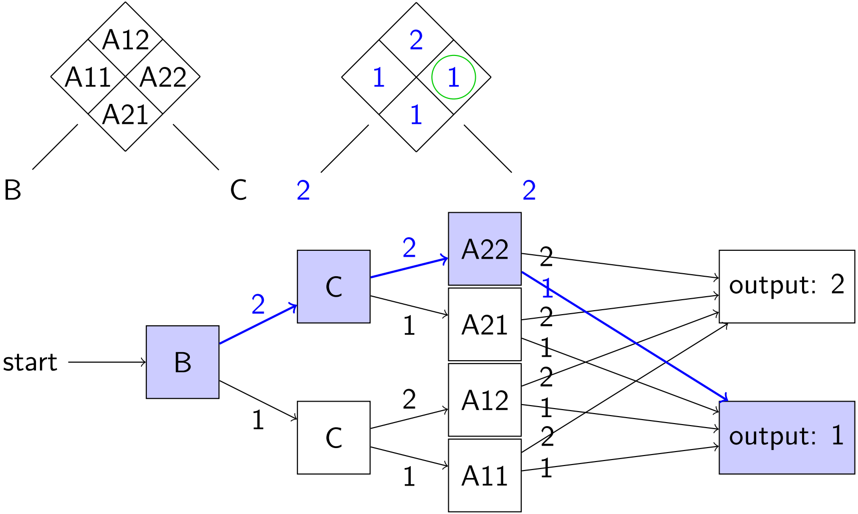
B
C


start
B
2
1
C
C
2
1
2
1
A22
A21
A12
A11
2
1
2
1
2
1
2
1
output: 2
output: 1


A12
A11
A22
A21
B
C
2
1
1
1
2
2
start
B
2
1
C
C
2
1
2
1
A22
A21
A12
A11
2
1
2
1
2
1
2
1
output: 2
output: 1


A12
A11
A22
A21
B
C
2
1
1
1
2
2
start
B
2
1
C
C
2
1
2
1
A22
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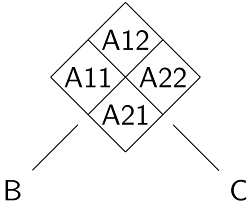
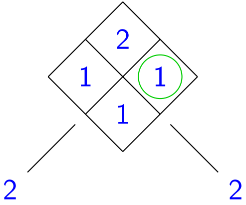
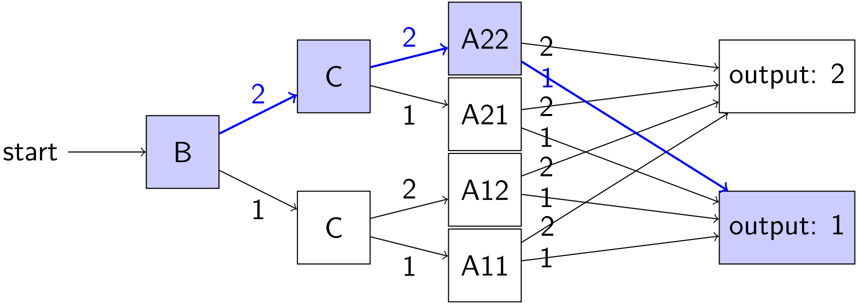
A12
A11
2
1
2
1
2
1
2
1
output: 2
output: 1







remember B

remember B, C

Pebbling game [Paterson Hewitt 1970]

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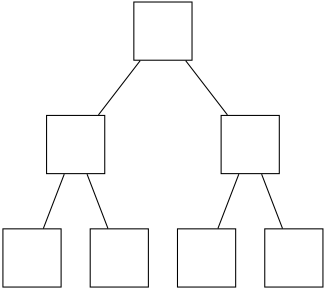

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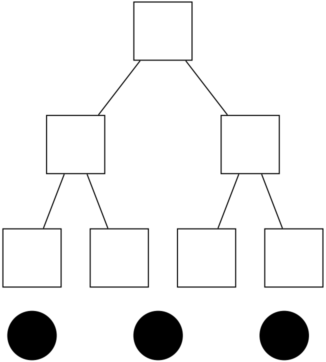

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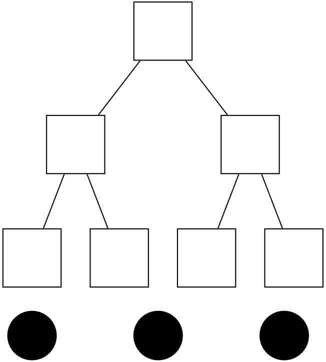

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Two kinds of move:

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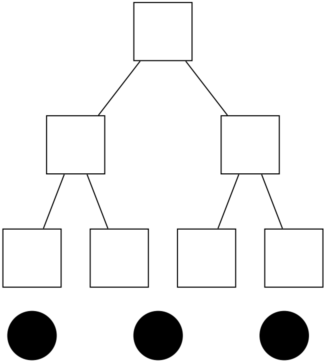

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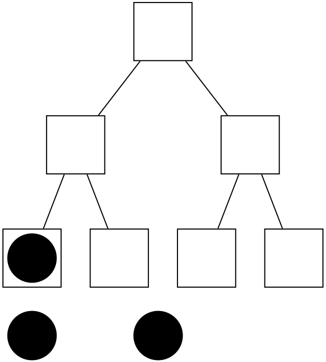

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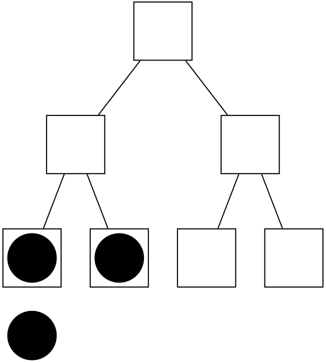

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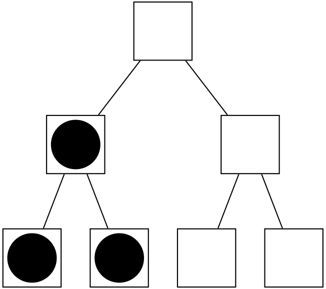

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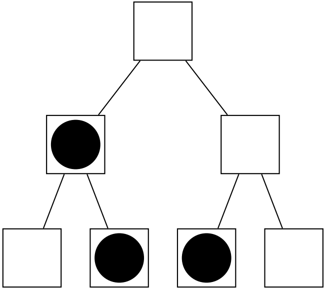

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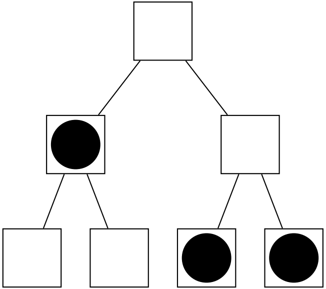

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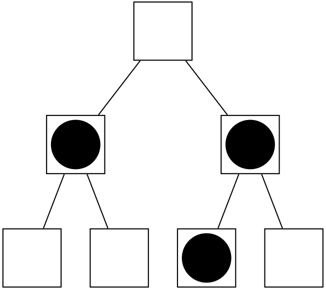

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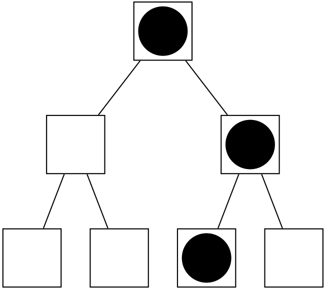

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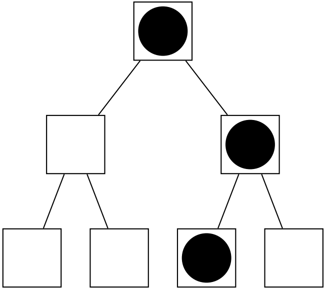

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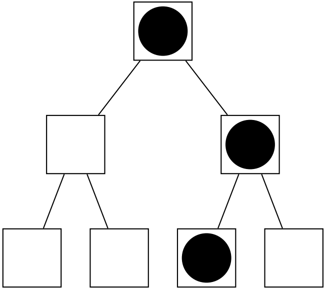

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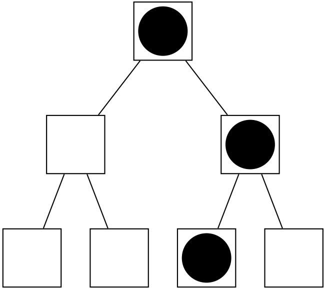

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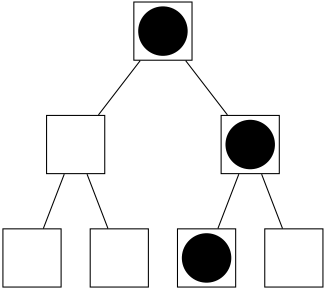

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Theorem: h pebbles are needed.
Conjecture (false): To solve TEP, a branching program needs $\Omega(k^h)$ states.

Conjecture (TEP $\notin$ L)

TEP can’t be solved by a uniform family of branching programs with $2^{O(h)}k^{O(1)}$ states.

Algorithm (pebbling)

The *pebbling algorithm* uses $\Theta((k+1)^h)$ states.

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Algorithm (new)

Our new algorithm uses $(O(\frac{k}{h}))^{2h+\epsilon}k^{\Theta(1)}$ states.

New algorithm defeats $\Omega(k^h)$ conjecture when $h \geq k^{1/2+\epsilon'}$, but is still not log space.

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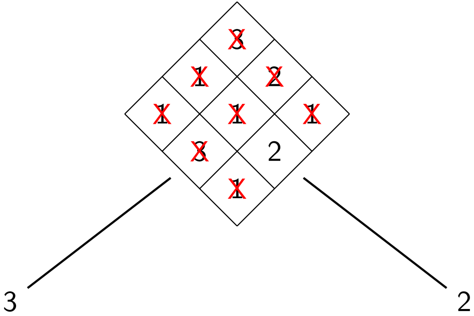
- the algorithm is *read-once*
- or the algorithm is *thrifty*: never reads an irrelevent piece of the input.



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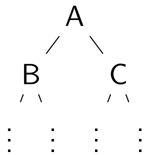

This rules out the following lower bound argument:

- At some point, you need to compute B.
- You need to remember B ($\log k$ bits) while computing C.
- So, every level of the tree adds $\log k$ bits you need to remember.

Bounded-width polynomial-size branching programs recognize exactly those languages in NC^1*.* [D. Barrington 1989]

Computing algebraic formulas using a constant number of registers. [M. Ben-Or, R. Cleve 1992]

Ring R
Inputs $x_1, \ldots, x_n \in R$
Work registers $r_1, \ldots, r_m \in R$

Reversible instructions:

- Example: $r_5 \leftarrow r_5 + r_4 \times x_1$.
- Inverse is $r_5 \leftarrow r_5 - r_4 \times x_1$.

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Definition

A sequence of reversible instructions *cleanly computes f into r_i* if, once it finishes:

- $r_i = \tau_i + f(x_1, \ldots, x_n)$
- all other registers are unchanged ($r_j = \tau_j$ for $j \neq i$)

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ℓ instuctions $\Rightarrow$ branching program with $(\ell + 1)|R|^m$ states.

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Cleanly compute $x_1 + x_2$ into r_1:

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Example

Cleanly compute $x_1 + x_2$ into r_1:

- $r_1 \leftarrow r_1 + x_1$ $[r_1 = \tau_1 + x_1]$
- $r_1 \leftarrow r_1 + x_2$ $[r_1 = \tau_1 + x_1 + x_2]$

Lemma: Multiplication

Suppose P_1 cleanly computes f_1 into r_1 and P_2 cleanly computes f_2 into r_2. Then we can cleanly compute $f_1 \times f_2$ into r_3 as follows:

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$r_3 \leftarrow r_3 - r_1 \times r_2$
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P_1	$\tau_1 + f_1$	τ_2	τ_3
$r_3 \leftarrow r_3 - r_1 \times r_2$			
P_2			
$r_3 \leftarrow r_3 + r_1 \times r_2$			
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$r_3 \leftarrow r_3 - r_1 \times r_2$	$\tau_1 + f_1$	τ_2	$\tau_3 - \tau_1 \times \tau_2 - f_1 \times \tau_2$
P_2			
$r_3 \leftarrow r_3 + r_1 \times r_2$			
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$r_3 \leftarrow r_3 + r_1 \times r_2$	$\tau_1 + f_1$	$\tau_2 + f_2$	$\tau_3 + \tau_1 \times f_2 + f_1 \times f_2$
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$r_3 \leftarrow r_3 - r_1 \times r_2$	τ_1	$\tau_2 + f_2$	$\tau_3 - \tau_1 \times \tau_2 + f_1 \times f_2$
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Cost: need to run P_1 and P_2 twice each. But: no memory needs to be reserved.

A formula for TEP

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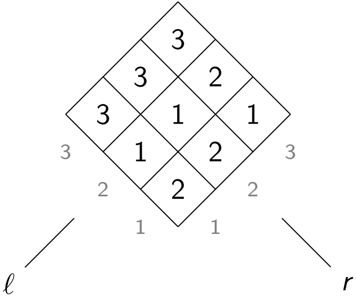
Suppose node v has children ℓ and r:



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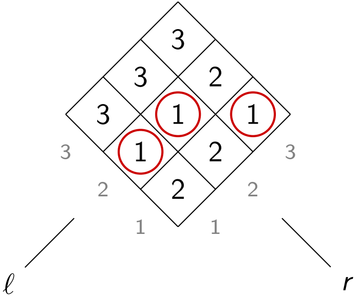

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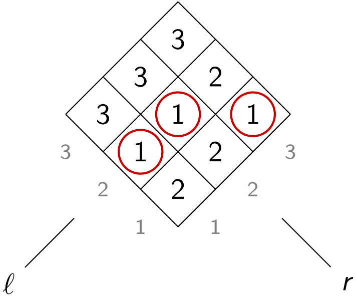

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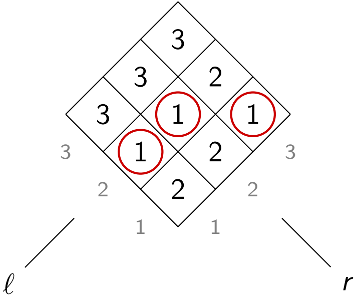

$$[v = 1] = \\ [\ell = 2] \times [r = 1] + [\ell = 2] \times [r = 2] + [\ell = 1] \times [r = 3]$$

Let f_v denote v's table. In general,

$$[v = x] = \sum_{(y,z) \in [k]^2} [f_v(y,z) = x] \times [\ell = y] \times [r = z]$$

First attempt

$$[v = x] = \sum_{(y,z)\in[k]^2} [f_v(y,z) = x] \times [\ell = y] \times [r = z]$$

Algorithm CheckNode(v, x, i)

Parameters: node v, value $x \in [k]$, target register i
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Worse than pebbling, which uses $\Theta((k+1)^h)$ states.

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$r_j \leftarrow r_j + [\ell = 1]$

$r_i \leftarrow r_i - r_j \times r_{j'}$

$r_{j'} \leftarrow r_{j'} + [r = 1]$

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$\cdots$

$\cdots$

$\cdots$

- for $(y, z) \in [k]^2$:
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$$
\begin{array}{llll}
r_j \leftarrow r_j + [\ell = 1] & r_j \leftarrow r_j + [\ell = 1] & r_j \leftarrow r_j + [\ell = 1] & \\
r_i \leftarrow r_i - r_j \times r_{j'} & r_i \leftarrow r_i - r_j \times r_{j'} & r_i \leftarrow r_i - r_j \times r_{j'} & \\
r_{j'} \leftarrow r_{j'} + [r = 1] & r_{j'} \leftarrow r_{j'} + [r = 2] & r_{j'} \leftarrow r_{j'} + [r = 3] & \cdots \\
r_i \leftarrow r_i + r_j \times r_{j'} & r_i \leftarrow r_i + r_j \times r_{j'} & r_i \leftarrow r_i + r_j \times r_{j'} & \cdots \\
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r_{j'} \leftarrow r_{j'} - [r = 1] & r_{j'} \leftarrow r_{j'} - [r = 2] & r_{j'} \leftarrow r_{j'} - [r = 3] & \\
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Running in parallel reduces to 4 recursive calls instead of $4k^2$. The catch: need $3k$ registers instead of 3.

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- "Hybrid encoding algorithm" interpolates between the two, and uses $(O(\frac{k}{h}))^{2h+\epsilon}k^{\Theta(1)}$ states.
    - Beats pebbling when $h \geq k^{1/2+\epsilon'}$.

Conclusion

- We present a new algorithm for TEP: first improvement over classic "pebbling" algorithm since the problem was introduced in 2010.
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Future work

- Improve the algorithm. (Better ways to compute *d*-ary products? We're not the first to want them.)
- Find new TEP lower bounds that apply to these algorithms. (Old lower bounds apply only to read-once or "thrifty" algorithms.)

Thanks!